Clean elements in polynomial rings

Pramod Kanwar, André Leroy, and Jerzy Matczuk

Abstract. The aim of the paper is to investigate the set, $Cl(R[x])$, of all clean elements in a polynomial ring $R[x]$. In particular, we present necessary and sufficient conditions for the set $Cl(R[x])$ to be a subring of $R[x]$ and show that if R is a clean 2-primal ring then $Cl(R[x])$ always forms a subring. We also show that the Köethe's problem has a positive solution if and only if $Cl(R[x])$ is a subring of $R[x]$ for any clean ring R such that $R/N(R)$ is reduced, where $N(R)$ denotes the nil radical of R.

Introduction

Throughout this article R will denote a unital associative ring. An element $a \in R$ is called clean if a can be expressed as $e + u$, where e is an idempotent and u is a unit of R. A ring R is clean if all its elements are clean. This notion was introduced by Nicholson [**N**] in relation to exchange rings. It was proved that every exchange ring is clean. The class of clean rings is quite large and includes, for example, semiperfect rings. For more information about clean rings we refer the reader to [**NZ**].

Han and Nicholson investigated the behavior of clean property under various ring extensions (see [**HN**]). In particular, they observed that if R is a clean ring then the power series ring $R[[x]]$ is always clean but the polynomial ring $R[x]$ is never clean, as the element x is never clean in $R[x]$. It is also clear that the clean elements of a ring need not form a subring, even in the case the ring is commutative (e.g. the clean elements of $\mathbb{Z}$ do not form a subring). Samei [**S**] proved that for a certain class of commutative rings, a ring is clean if and only if the set of all its clean elements is closed under addition.

The aim of the paper is to determine the set $Cl(R[x])$ of all clean elements of $R[x]$. We show that $Cl(R[x]) = Cl(R) + B(R)[x]x$, where $B(R)$ denotes the prime radical of R, if and only if R is 2-primal, that is, when $R/B(R)$ is a reduced ring (see Theorem 1.5). This implies that if R is a 2-primal clean ring, then the set $Cl(R[x])$ is a subring of $R[x]$. In Theorem 2.14 we give necessary and sufficient conditions for $Cl(R[x])$ to be a subring. Finally, in Theorem 2.15, we show that the positive solution to the Köthe's problem is equivalent to the statement that the set

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$Cl(R[x])$ is a subring of $R[x]$ for any clean ring R such that $R/N(R)$ is reduced, where $N(R)$ denotes the nil radical of R. We also give some examples.

Throughout the paper $Cl(R), U(R)$, and $E(R)$ will stand for the sets of all clean elements, units, and idempotents of the ring R, respectively and $J(R)$, $N(R)$, and $B(R)$ will denote the Jacobson radical, the upper nil radical, and the prime radical of R.

1. 2-primal coefficient ring

Recall that a ring is called abelian if its idempotents are central. Note that every reduced ring is abelian.

We will frequently use the following observation.

Lemma 1.1. (Lemma 1, [**KLM**]) Let $e(x) = e_0 + e_1x \ldots + e_nx^n$ be an idempotent in $R[x]$. If e_0 commutes with all e_i, $0 \leq i \leq n$, then $e(x) = e_0$. In particular, if R is abelian then $E(R[x]) = E(R)$.

We begin with elementary lemmas.

Lemma 1.2. *Let R be a ring. Suppose that $u \in U(R)$ and $a \in J(R)$. Then $u + a \in U(R)$. In particular, for any clean element $c \in R$, the element $c + a$ is also clean, that is, $Cl(R) + J(R[x]) \subseteq Cl(R[x])$.*

Proof. Since $a \in J(R)$, we have $u^{-1}a \in J(R)$. Thus $u + a = u(1 + u^{-1}a)$ is invertible being a product of invertible elements. The second part of the lemma is a straightforward consequence of the first statement. $\square$

The following example shows that the inclusion $Cl(R) + J(R[x]) \subseteq Cl(R[x])$ can be strict, in general.

Example 1.3. Let $R = M_2(\mathbb{Q})$ and $a = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \in R$. Since $\det(ax + b) \in \mathbb{Q} \setminus \{0\}$, the element $ax + b$ is invertible in $M_2(\mathbb{Q}[x]) = R[x]$, so it is clean. Clearly $J(R) = 0$.

Observe that if $w(x) = w_0 + w_1x \ldots + w_nx^n \in R[x]$ is invertible (respectively, is an idempotent or a clean element) in $R[x]$, then so is the element w_0 in R. Hence we have the following lemma.

Lemma 1.4. *If $r \in R$ is clean in $R[x]$, then it is clean in R. In particular, $Cl(R[x]) \cap R = Cl(R)$.*

The following theorem gives a characterization of 2-primal rings in terms of clean polynomials. Recall that a ring R is called 2-primal if $R/B(R)$ is a reduced ring, where $B(R)$ denotes the prime radical of R. Equivalently, R is 2-primal if and only if $B(R)$ is equal to the set of all nilpotent elements of R. It is well-known that the ring R is 2-primal if and only if every minimal prime ideal P of R is completely prime, that is, R/P is a domain.

Theorem 1.5. *For any ring R, the following are equivalent:*

(i) *R is 2-primal;*
(ii) *$R[x]$ is 2-primal;*
(iii) *$Cl(R[x]) = Cl(R) + B(R)[x]x$;*
(iv) *$U(R[x]) = U(R) + B(R)[x]x$.*

PROOF. The equivalence (i)$\Leftrightarrow$(ii) is easy and well-known.

(i)$\Rightarrow$(iii) Let $C = Cl(R) + B(R)[x]x$. Since $B(R)[x] = B(R[x]) \subseteq J(R[x])$, Lemma 1.2 shows that $C \subseteq Cl(R[x])$.

Let $c(x) = \sum_{i=0}^{n} c_i x^i \in Cl(R[x])$. We then have $c_0 \in Cl(R)$. Let P be a minimal prime ideal of R. Then the canonical image $\bar{c}(x)$ of $c(x)$ in $(R/P)[x]$ is also clean. Since R is 2-primal, R/P is a domain. Hence $E((R/P)[x]) = \{0, 1\}$, $U((R/P)[x]) = U(R/P)$, and $Cl(R/P)[x] \subseteq Cl(R/P)$. This means that, for $i \geq 1$, $c_i \in P$ and hence $c(x) \in R + P[x]x$. Since this is true for any minimal prime ideal P, we obtain that $c(x) \in R + B(R)[x]x$. This means that $c(x) \in Cl(R) + B(R)[x]x$ as $c_0 \in Cl(R)$.

(iii)$\Rightarrow$ (iv) Using Lemma 1.2 and the statement (iii) we get $U(R) + B(R)[x]x \subseteq U(R[x]) \subseteq Cl(R[x]) = Cl(R) + B(R)[x]x$. Since the independent term of an invertible polynomial is invertible in R, we conclude that $U(R[x]) \subseteq U(R) + B(R)[x]x$.

(iv)$\Rightarrow$(i) If $a \in R$ is such that $a^n = 0$ then $1 + xa \in U(R[x])$ and (iv) shows that $a \in B(R)$, that is, R is 2-primal. $\square$

REMARKS 1.6. (a) Using the equivalence (i)$\Leftrightarrow$(ii) in the above theorem it is standard to see that all statements of Theorem 1.5 remain valid if we replace the single indeterminate x by a finite set of commuting indeterminates.

(b) Since $B(R[x, x^{-1}]) = B(R)[x, x^{-1}]$, it is easy to check that R is 2-primal if and only if the Laurent polynomial ring $R[x, x^{-1}]$ is 2-primal. Obviously the analogous of statements (iii) and (iv) of Theorem 1.5 are not true in the case of $R[x, x^{-1}]$.

(c) Note that in the Laurent polynomial ring $(\mathbb{Z}/6\mathbb{Z})[x, x^{-1}]$ we have $(2x + 3x^{-1})(3x - 4x^{-1}) = 1$. This shows that $U(R[x, x^{-1}])$ is not necessarily homogeneous even if R is a commutative reduced ring.

COROLLARY 1.7. *For a ring R the following are equivalent:*

(i) *R is reduced;*
(ii) *$U(R[x]) = U(R)$;*
(iii) *$Cl(R[x]) = Cl(R)$.*

PROOF. The implication (i)$\Rightarrow$(iii) is given by Theorem 1.5.

Since $U(R[x]) \subseteq Cl(R[x])$, (iii) implies (ii).

Finally suppose that (ii) holds. If $a \in R$ is nilpotent then $1 - ax \in U(R[x]) = U(R)$, that is, $a = 0$ and R is a reduced ring. $\square$

As a another consequence of Theorem 1.5 we also get the following corollary.

COROLLARY 1.8. *Suppose R is 2-primal. Then $Cl(R[x])$ is a subring of $R[x]$ if and only if $Cl(R)$ is a subring of R.*

Since commutative rings are 2-primal, we have the following corollary.

COROLLARY 1.9. *Suppose R is a commutative clean ring. Then the set $Cl(R[x])$ is a subring of $R[x]$.*

2. When $Cl(R[x])$ forms a subring of $R[x]$

We have seen that if R is a 2-primal clean ring, then the set $Cl(R[x])$ is always a subring of $R[x]$. We begin this section with an example showing that the set of clean elements of a polynomial ring over a clean ring does not have to be a subring in general. For this we need the following lemma.

LEMMA 2.1. *The set of all clean elements of the matrix ring $M_n(R)$, $n \geq 2$, forms a subring if and only if $M_n(R)$ is a clean ring.*

PROOF. Let $\{e_{ij} \mid 1 \leq i, j \leq n\}$ denotes the set of matrix units in $M_n(R)$. Since nilpotent elements and idempotents are clean, the elements $e_{1n}r, e_{ij}$, with $r \in R$ and $1 \leq i, j \leq n$, are clean. Moreover these elements generate $M_n(R)$ as a ring. This gives the lemma. □

EXAMPLE 2.2. Let K be a field and $n \geq 2$. Then $M_n(K)$ is a clean ring and the set of clean elements of $K[x]$ is equal K. Let us consider $T = M_n(K[x]) \simeq M_n(K)[x]$. Then T is not clean as a polynomial ring is never clean. Thus, by the above lemma, the set $Cl(M_n(K[x]))$ does not form a subring of the matrix ring $M_n(K[x])$ as well of the polynomial ring $M_n(K)[x]$.

Before we give more results on clean elements forming a subring, we give some results on nilpotent polynomials. This will also lead to a relationship between the description of the set of clean elements of a polynomial ring and the Köthe's problem.

LEMMA 2.3. *Let $a = e + u$, for some idempotent e and a unit u of R. If au^{-1} is a nilpotent element, then $e = 1$.*

PROOF. Suppose that $au^{-1} = eu^{-1} + 1$ is a nilpotent element. Then $eu^{-1} = au^{-1} - 1$ is an invertible element of R such that $(1-e)eu^{-1} = 0$, that is, $e = 1$. □

LEMMA 2.4. *Let $p(x) = a_0 + a_1x + \ldots + a_nx^n \in R[x]$ be such that a_0v is a nilpotent element of R, for every $v \in U(R)$. Then $p(x)$ is a clean element of $R[x]$ if and only if $1 - p(x) \in U(R[x])$.*

PROOF. It is enough to prove the necessity of the condition. Suppose that $p(x)$ is clean. We can write $p(x) = e(x) + u(x)$, where $e(x) = e_0 + \ldots + e_mx^m \in E(R[x])$ and $u(x) = u_0 + \ldots + u_mx^m \in U(R[x])$. Then $a_0 = e_0 + u_0$, where $e_0 \in E(R)$ and $u_0 \in U(R)$. By the assumption, the element $a_0u_0^{-1}$ is nilpotent. Thus, Lemma 2.3 implies that $e_0 = 1$ and Lemma 1.1 then shows that $e(x) = 1$. Therefore, $1 - p(x) = -u(x) \in U(R[x])$. □

PROPOSITION 2.5. *Let $l > 0$ be a positive integer and $f(x) \in R[x]$ be a polynomial of degree $\deg f(x) < l$. Then the following conditions are equivalent:*

(i) $x^l f(x) \in Cl(R[x])$;
(ii) $f(x)$ *is nilpotent;*
(iii) $1 - x^l f(x) \in U(R[x])$.

PROOF. (i)⇒(ii) Suppose $x^l f(x) \in Cl(R[x])$. Notice that the element $A = \sum_{i=0}^{\infty} x^{li} f(x)^i$ is well defined in the power series ring $R[[x]]$ and A is the inverse of $1 - x^l f(x)$. However, Lemma 2.4 gives that $1 - x^l f(x)$ is invertible in $R[x]$, that is, $A \in R[x]$. Since, for any $i \geq 1$, the degree of $x^{li} f(x)^i$ is strictly smaller than the degree of any monomial appearing in $x^{(l+1)i} f(x)^{i+1}$, we conclude that there exists $n \in \mathbb{N}$ such that $x^{li} f(x)^i = 0$, for all $i \geq n$. The desired conclusion then follows.

Implications (ii)⇒(iii) and (iii)⇒(i) are obvious and hold without the assumption on l. □

Corollary 2.6. *For any $a \in R$ and $n \geq 1$, the monomial $ax^n \in R[x]$ is clean if and only if $ax^n - 1$ is invertible if and only if a is nilpotent.*

The following example shows that the equivalence in Proposition 2.5 is false if $l = \deg f(x)$.

Example 2.7. Let $R = M_2(\mathbb{F}_2)$ be the ring of 2×2 matrices over the field with two elements. Define elements $a = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in R$. It is easy to check that $(1 + ax + bx^2)(1 + ax + (1 + b)x^2) = 1$. This shows that the polynomial $x(a + bx)$ is clean. It can be easily seen that $a + bx$ is not nilpotent.

Remark 2.8. Corollary 2.6 implies that $x \in R[x]$ is never clean. Hence, as is well known, $R[x]$ is never a clean ring.

Recall that $N(R)$ denotes the nil radical of R. In general the description of all clean as well as invertible elements of a polynomial ring seems to be a difficult problem as the following theorem shows.

Theorem 2.9. *For any ring R, the following conditions are equivalent:*
(i) $N(R)[x] = J(R[x])$;
(ii) $1 - N(R)[x] \subseteq U(R[x])$;
(iii) $N(R)[x] \subseteq Cl(R[x])$.

Proof. Clearly (i)⇒(ii)⇒(iii).
(iii)⇒(i) It is known that the first statement is equivalent to $N(R)[x] \subseteq J(R[x])$ (then the equality holds). Let $g(x) \in N(R)[x]$. Then, for any $f(x) \in R[x]$, $g(x)f(x) \in N(R)[x] \subseteq Cl(R[x])$ and Lemma 2.4 shows that $1 - g(x)f(x) \in U(R[x])$, that is, $g(x) \in J(R[x])$. □

Lemma 2.10. *Suppose $Cl(R)$ is a subring of R. Then:*
(i) *The factor ring $Cl(R)/N(R)$ is reduced if and only if $R/N(R)$ is a reduced ring;*
(ii) *If $R/N(R)$ is a reduced ring, then $Cl(R[x]) \subseteq Cl(R) + N(R)[x] \subseteq Cl(R)[x]$.*

Proof. (i) If the paper $R/N(R)$ is reduced then its subring $Cl(R)/N(R)$ is also reduced.
Conversely suppose that $Cl(R)/N(R)$ is a reduced ring and let $a + N(R) \in R/N(R)$ be a nilpotent elenent. This means that there exists $l \in \mathbb{N}$ such that $a^l \in N(R)$, that is, a is a nilpotent element and so $a \in Cl(R)$. The hypothesis then gives $a \in N(R)$ and shows that $R/N(R)$ is a reduced ring.
(ii) Suppose now that $Cl(R)$ is a subring of R and the ring $R/N(R)$ is reduced. Let $c(x)$ be an element in $Cl(R[x])$. Then the natural image $\bar{c}(x)$ of $c(x)$ in $R[x]/N(R)[x] \cong R/N(R))[x]$ is a clean element. Thus, by Corollary 1.7, $\bar{c}(x) \in R/N(R)$. This means that $c(x) \in Cl(R) + N(R)[x]$ and yields $Cl(R)[x] \subseteq Cl(R) + N(R)[x]$. The inclusion $Cl(R) + N(R)[x] \subseteq Cl(R)[x]$ is clear as $N(R) \subseteq Cl(R)$. □

In the following proposition we gather some necessary conditions for $Cl(R[x])$ to be a subring of $R[x]$.

Proposition 2.11. *Suppose that $Cl(R[x])$ is a subring of $R[x]$. Then:*
(i) *$Cl(R)$ is a subring of R;*

(ii) $Cl(R[x]) = Cl(R) + U(R[x])$;
(iii) *The set of nilpotent elements of* R *is an ideal of* $Cl(R)$;
(iv) $R/N(R)$ *is a reduced ring;*
(v) $Cl(R[x]) \subseteq Cl(R) + N(R)[x] \subseteq Cl(R)[x]$.

PROOF. (i) $Cl(R)$ is a subring of R, by Lemma 1.4.

(ii) Let $f(x) = \sum_{i=0}^{n} f_i x^i \in Cl(R[x])$. Since $f_0 \in Cl(R) \subseteq Cl(R[x])$, we also have $f(x) - f_0 \in Cl(R[x])$. Let us write $f(x) - f_0 = e(x) + u(x)$ with $e(x) \in E(R[x])$ and $u(x) \in U(R[x])$. We must then have $e(0)^2 = e(0) = -u(0) \in U(R)$. This shows that $e(0) = 1$ and Lemma 1.1 implies that $e(x) = 1$. Thus we get $f(x) = f_0 + 1 + u(x) \in Cl(R) + U(R[x])$. This shows that $Cl(R[x]) \subseteq Cl(R) + U(R[x])$. The reverse inclusion is an obvious consequence of the hypothesis that $Cl(R[x])$ forms a subring.

(iii) Let $a, b \in R$ be nilpotent elements and $r \in Cl(R)$. Then the elements ax, bx, being nilpotent, are clean. Thus, by assumption $(a-b)x, rax, arx = (ax)r \in Cl(R[x])$ and Corollary 2.6 implies that the set of all nilpotent elements of R is an ideal of $Cl(R)$.

(iv) The statement (iii) together with the inclusion $N(R) \subseteq Cl(R)$ imply that $N(R)$ is equal to the set of all nilpotent elements of R, that is, $R/N(R)$ is a reduced ring.

(v) Statements (i) and (iv) together with Lemma 2.10 directly yield the result. □

A matrix ring $M_n(R)$ is never reduced if $n \geq 2$. Thus, by the above proposition, we have the following corollary (compare with Example 2.2).

COROLLARY 2.12. *Let* $n \geq 2$. *Then, for any ring* R, *the set* $Cl(M_n(R)[x])$ *does not form a subring of* $M_n(R)[x]$.

Let us give an example showing that the inclusion $Cl(R[x]) \subseteq Cl(R)[x]$ does not always hold.

EXAMPLE 2.13. Let $R = M_2(\mathbb{Z})$. Consider

$$a = \begin{bmatrix} -7 & -3 \\ 12 & 5 \end{bmatrix} \quad b = \begin{bmatrix} 12 & 5 \\ 0 & 0 \end{bmatrix} \quad c = a^{-1} = \begin{bmatrix} 5 & 3 \\ -12 & -7 \end{bmatrix} \quad d = \begin{bmatrix} 0 & -5 \\ 0 & 12 \end{bmatrix}.$$

It is easy to check that $(a + bx)(c + dx) = 1 = (c + dx)(a + bx)$. Thus $a + bx$ is invertible (so clean). By Example 4.5 of [**KL**], the matrix b is not clean. This shows that the inclusion $Cl(R[x]) \subseteq Cl(R)[x]$ does not hold.

The above results lead to the following theorem which characterizes when $Cl(R[x])$ forms a subring of $R[x]$.

THEOREM 2.14. *Let* R *be any ring. Then the following conditions are equivalent:*

(i) *The set* $Cl(R[x])$ *forms a subring of* $R[x]$;
(ii) $Cl(R)$ *is a subring of* R *and* $Cl(R[x]) = Cl(R) + N(R)[x]$.

PROOF. (i)⇒(ii) Suppose $Cl(R[x])$ is a subring of $R[x]$. Then any polynomial with nilpotent coefficients is clean. Hence $Cl(R) + N(R)[x] \subseteq Cl(R[x])$. The reverse inclusion is given by Proposition 2.11 (v). The statement (i) from the same proposition shows that $Cl(R)$ is a subring of R, that is, (ii) holds.

The implication (ii)⇒(i) is a tautology. This completes the proof of the theorem. □

In the context of Proposition 2.11, one can wonder whether in the above theorem one could add as an equivalent statement that $Cl(R)/N(R)$ is a reduced ring. It turns out, as the following theorem shows, that this is equivalent to a positive solution of the Köthe problem. We state this result in case of clean rings to make the statements easier.

THEOREM 2.15. *The following conditions are equivalent:*
- (i) *The Köthe's problem has a positive solution;*
- (ii) *For any clean ring R, the set $Cl(R[x])$ forms a subring of $R[x]$ if and only if $R/N(R)$ is a reduced ring;*
- (iii) *For any clean ring R, the set $Cl(R[x])$ forms a subring of $R[x]$, provided $R/N(R)$ is a reduced ring.*

PROOF. (i)⇒(ii) Suppose R is a clean ring. Proposition 2.11 shows that if $Cl(R[x])$ is a subring then $R/N(R)$ is a reduced ring. Hence we need only prove that if the Köthe's problem has a positive solution and $R/N(R)$ is reduced, then $Cl(R[x])$ is a subring of $R[x]$. Under these hypotheses, Corollary 1.7 shows that $Cl((R[x])/(N(R)[x])) = Cl((R/N(R))[x]) = Cl(R/N(R))$. Therefore $Cl(R[x]) \subseteq Cl(R) + N(R)[x]$. Recall that the Köthe's problem has a positive solution if and only if $J(R[x]) = N(R)[x]$, for any ring R. Hence, by (i) and Lemma 1.2, $Cl(R) + N(R)[x] \subseteq Cl(R[x])$, for any ring R. Thus, by the above we get $Cl(R[x]) = Cl(R) + N(R)[x]$ and Theorem 2.14 shows that $Cl(R[x])$ forms a subring of R.

The implication (ii)⇒(iii) is a tautology.

(iii)⇒(i) Suppose (iii) holds. It is well known (Theorem 6, [**K**]) that the Köthe's problem has a positive solution if and only if it has positive solution for algebras over fields. Henceforth let T be a nil algebra over a field K. Our aim is to show that $J(T[x]) = T[x]$. Let T^* denote the K-algebra with unity adjoined to T. Then $N(T^*) = T$, $T^*/N(T^*) = K$. Moreover, by the first part of Lemma 1.2, T^* is a clean ring. Thus, the statement (iii) and Theorem 2.14 imply that $Cl(T^*[x]) = T^* + T[x]$. This means, in particular, that $T[x] = N(T^*)[x] \subseteq Cl(T^*)[x]$ and Theorem 2.9 yields that $J(T[x]) = T[x]$. □

We conclude the paper with the simple observation that for any ring R the set of clean elements of R[[x]] is precisely $Cl(R) + R[[x]]$. We leave the easy proof of this fact to the reader.

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Ohio University, Department of Mathematics, Zanesville, Ohio, USA
E-mail address: kanwar@ohio.edu

Université d'Artois, Faculté Jean Perrin, Rue Jean Souvraz 62 307 Lens, France
E-mail address: leroy@euler.univ-artois.fr

Institute of Mathematics, Warsaw University, Banacha 2, 02-097 Warsaw, Poland
E-mail address: jmatczuk@mimuw.edu.pl